MATHEMATICAL METHODS FOR SOLUTION OF NONSTATIONARY RADIANT—CONVECTIVE HEAT-EXCHANGE PROBLEMS

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A new analytical method is developed for solution of nonstationary radiant—convective heat-exchange problems in a moving, viscous, absorbing, radiating, and anisotropically scattering medium. The method is applied to solution of the problem for a semiinfinite body.

The simultaneous transfer of energy by thermal conductivity, convection, and radiation is described by a complex system of equations including the differential equations of continuity and motion of the medium, the integrodifferential equation of energy transfer, and the integral radiation equations [1-3]. Solution of this system is exceedingly difficult. However, in the case of a viscous incompressible medium the system can be separated into gasdynamic and thermal systems [4, 5]. The closed system of five equations describing the gasdynamic problem thus obtained permits determination of five unknown functions $\rho(M, t)$, $p(M, t)$, and $V(M, t)$. Assuming that this problem has been solved beforehand, we turn to the nonlinear integrodifferential equation of the form [2]

$$\frac{dT}{dt} - a(M, t)\nabla^2 T = \frac{\sigma_0 \alpha(M, t)}{C_p \rho(M, t)} \left\{ \int_F A(N, t)\Omega_*(M, N)[T^4(N, t) - T^4(M, t)]\, dF_N + 4\int_V \alpha(P, t)\chi_*(M, P)[T^4(P, t) - T^4(M, t)]\, dV_p \right\} + \frac{F(M, t)}{C_p \rho(M, t)}, \quad (1)$$

where

$$F(M, t) = \frac{dp}{dt} + 2\mu \operatorname{Diss} f(\mathbf{V}) + \rho q. \quad (2)$$

The resolvents $\Omega_*(M, N)$ and $\chi_*(M, P)$ appearing in Eq. (1) are defined by expressions presented in [1].

To solve Eq. (1) it is necessary to specify boundary conditions. These consist of an initial temperature distribution in the medium

$$T(M, 0) = T_i(M) \quad (3)$$

and the thermal regime at the boundary of the radiating system. If the boundary surface of the radiating system is maintained at a specified temperature, then

$$T(M, t)|_{M \in F} = T_W(M, t). \quad (4)$$

For the case of radiant—convective heat transfer during motion of a viscous, absorbing, radiating, and anisotropically scattering medium, the boundary condition may be represented in the form

$$\lambda \frac{\partial T}{\partial n}\bigg|_{M \in F} = -\alpha_{em}[T(N, t) - T_c(N, t)] - \sigma_0 A[T^4(N, t) - T_c^4(N, t)]. \quad (5)$$

We will introduce the dimensionless variables $x = lx^*$, $y = ly^*$, $z = lz^*$, $t = (l^2/a_0)t^*$, $T = T_0T^*$, $a = a_0a^*$, $\alpha = k_0\alpha^*$, $\rho = \rho_0\rho^*$, $q = q_0q^*$, $p = \Delta p p^*$, $\mathbf{V} = V_0\mathbf{V}^*$, $\Omega_* = (1/l^2)\Omega_*^*$, $\chi_* =$

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$(1/l^2)\chi_*^*$, where l is the characteristic dimension and V_0 is the characteristic velocity of medium motion.

In these new variables, if we omit the asterisks, Eqs. (1)-(5) take on the form

$$\frac{\partial T}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla T)-a(M,t)\nabla^2 T=\frac{\mathrm{Bu}\,\mathrm{Pe}}{\mathrm{Bo}}\cdot\frac{\alpha(M,t)}{\rho(M,t)}\times$$
$$\times\left\{\int_F A(N,t)\Omega_*(M,N)[T^4(N,t)-T^4(M,t)]\,dF_N+4\,\mathrm{Bu}\times\right.$$
$$\left.\times\int_V \alpha(P,t)\chi_*(M,P)[T^4(P,t)-T^4(M,t)]\,dV_p\right\}+\frac{F(M,t)}{\rho(M,t)}, \tag{6}$$

$$T(M,0)=T_{\mathrm{i}}(M), \tag{7}$$

$$T(M,t)|_{M\in F}=T_W(M,t), \tag{8}$$

$$\left.\frac{\partial T}{\partial n}\right|_{M\in F}=-\mathrm{Bi}\,[T(N,t)-T_{\mathrm{c}}(N,t)]-\mathrm{Bi}_{\mathrm{p}}\,[T^4(N,t)-T_{\mathrm{c}}^4(N,t)], \tag{9}$$

where

$$F(M,t)=\frac{\mathrm{Eu}\,\mathrm{Ec}}{2}\left[\frac{\partial p}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla p)\right]+\frac{2\mathrm{Ec}\,\mathrm{Pe}}{\mathrm{Re}}\,\mathrm{Diss}\,f(\mathbf{V})+\mathrm{Os}\,\rho(M,t)\,q(M,t), \tag{10}$$

and the dimensionless complexes Pe, Bu, Bo, Bi, Bi_{p}, Eu, Ec, Re and Os are the Peclét, Burger, Boltzmann, Biot, radiation Biot, Euler, Eckert, Reynolds, and Ostrogradskii numbers [6].

We will seek a solution of Eqs. (6)-(9) in the form of a series in powers of the Burger number Bu

$$T(M,t)=\sum_{n=0}^{\infty}\mathrm{Bu}^n T_n(M,t). \tag{11}$$

Substituting Eq. (11) in Eq. (6) and equating coefficients of identical powers of Bu, we arrive at the following infinite sequence of differential equations with partial derivatives

$$\frac{\partial T_0}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla T_0)-a(M,t)\nabla^2 T_0=\frac{F(M,t)}{\rho(M,t)}, \tag{12}$$

$$\frac{\partial T_1}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla T_1)-a(M,t)\nabla^2 T_1=\frac{\mathrm{Pe}}{\mathrm{Bo}}\,\frac{\alpha(M,t)}{\rho(M,t)}\int_F A(N,t)\Omega_*(M,N)[J_0(N,t)-J_0(M,t)]\,dF_N, \tag{13}$$

$$\frac{\partial T_2}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla T_2)-a(M,t)\nabla^2 T_2=\frac{\mathrm{Pe}}{\mathrm{Bo}}\,\frac{\alpha(M,t)}{\rho(M,t)}$$

$$\left\{\int_F A(N,t)\Omega_*(M,N)[J_1(N,t)-J_1(M,t)]\,dF_N+4\int_V \alpha(P,t)\chi_*(M,P)[J_0(P,t)-J_0(M,t)]\,dV_p\right\}, \tag{14}$$

. . . .

$$\frac{\partial T_n}{\partial t}+\mathrm{Pe}(\mathbf{V},\nabla T_n)-a(M,t)\nabla^2 T_n=\frac{\mathrm{Pe}}{\mathrm{Bo}}\,\frac{\alpha(M,t)}{\rho(M,t)}\times\left\{\int_F A(N,t)\Omega_*(M,N)[J_{n-1}(N,t)-J_{n-1}(M,t)]\,dF_N\right.$$

$$\left.+4\int_V \alpha(P,t)\chi_*(M,P)[J_{n-2}(P,t)-J_{n-2}(M,t)]\,dV_p\right\}, \tag{15}$$

. . . .

where

$$J_n(M, t) = \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{l=0}^{n-m} T_k(M, t) T_{m-k}(M, t) T_l(M, t) T_{n-m-l}(M, t). \quad (16)$$

Similar substitution of Eq. (11) in Eqs. (7)-(9) leads to boundary conditions for each of the functions $T_n(M, t)$:

initial conditions

$$T_0(M, 0) = T_i(M), \; T_1(M, 0) = 0, \ldots, \; T_n(M, 0) = 0, \ldots, \quad (17)$$

boundary conditions in the case of Eq. (8)

$$T_0(M, t)|_{M \in F} = T_W(M, t), \; T_1(M, t)|_{M \in F} = 0, \ldots, \; T_n(M, t)|_{M \in F} = 0, \ldots, \quad (18)$$

in the case of Eq. (9)

$$\left.\frac{\partial T_0}{\partial n}\right|_{M \in F} = -\mathrm{Bi}\,[T_0(N, t) - T_c(N, t)] - \mathrm{Bi_p}\,[J_0(N, t) - T_c^4(N, t)], \quad (19)$$

$$\left.\frac{\partial T_1}{\partial n}\right|_{M \in F} = -\mathrm{Bi}\,T_1(N, t) - \mathrm{Bi_p} J_1(N, t), \quad (20)$$

. . . .

$$\left.\frac{\partial T_n}{\partial n}\right|_{M \in F} = -\mathrm{Bi}T_n(N, t) - \mathrm{Bi_p} J_n(N, t). \quad (21)$$

. . . .

It should be noted that each of the equations of the infinite sequence Eqs. (12)-(15) is itself a linear inhomogeneous partial differential equation with a known right-hand side. Thus, the proposed method for solution of nonstationary radiant-convective heat-exchange problems reduces the solution of the nonlinear integrodifferential equation (6) to solution of a series of linear differential equations, each of which is a further refinement of the solution obtained on the basis of preceding equations of the sequence.

If the medium is immobile ($V \equiv 0$) and nonradiating [$\alpha(M, t) \equiv 0$], then from Eqs. (12)-(16) and (17), (19)-(21) we have, assuming p = const for simplicity,

$$\frac{\partial T_0}{\partial t} = a\nabla^2 T_0 + \mathrm{Os}\,q, \quad (22)$$

$$T_0(M, 0) = T_i(M), \quad (23)$$

$$\left.\frac{\partial T_0}{\partial n}\right|_{M \in F} = \mathrm{Bi}\,[T_0(N, t) - T_c(N, t)] + \mathrm{Bi_p}\,[T_0^4(N, t) - T_c^4(N, t)]. \quad (24)$$

The following functions $T_n(M, t)$ ($n = 1, 2, \ldots$) as can easily be seen, are identically equal to zero. The solution of Eqs. (22)-(24) presents no difficulties [3]. In particular, for a semiinfinite medium (assuming a = const) we have

$$T_0(x, t) = \int_0^\infty T_i(\xi) G(x, t; \xi, 0)\, d\xi + \mathrm{Os} \int_0^t d\tau \int_0^\infty q(\xi, \tau) G(x, t; \xi, \tau)\, d\xi + \int_0^t f(z, \tau) G(x, t; 0, \tau)\, d\tau, \quad (25)$$

where

$$G(x, t; \xi, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \left\{\exp\left[-\frac{(\xi - x)^2}{4(t-\tau)}\right] + \exp\left[-\frac{(\xi + x)^2}{4(t-\tau)}\right]\right\}; \quad (26)$$

$$f(z, t) = \mathrm{Bi}\,[T_c(t) - z] + \mathrm{Bi_p}\,[T_c^4(t) - z^4]. \; z(t) = T(0, t); \quad (27)$$

$$z(t) + \mathrm{Bi} \int_0^t z(\tau) G(0, t; 0, \tau)\, d\tau + \mathrm{Bi_p} \int_0^t z^4(\tau) G(0, t; 0, \tau)\, d\tau =$$

$$= \int_0^\infty T_i(\xi) G(0, t; \xi, 0)\, d\xi + \mathrm{Os} \int_0^t d\tau \int_0^\infty q(\xi, \tau) G(0, t; \xi, \tau)\, d\xi$$

$$+ \mathrm{Bi} \int_0^t T_c(\tau) G(0, t; 0, \tau)\, d\tau + \mathrm{Bi_p} \int_0^t T_c^4(\tau) G(0, t; 0, \tau)\, d\tau. \tag{28}$$

If $T_i(x) = T_i = \text{const}$, $T_c(t) = T_c = \text{const}$, $q(\xi, \tau) = q_0 = \text{const}$, then Eq. (28) simplifies, taking on the form

$$z(t) + \frac{\mathrm{Bi}}{\sqrt{\pi}} \int_0^t \frac{z(\tau)}{\sqrt{t-\tau}}\, d\tau + \frac{\mathrm{Bi_p}}{\sqrt{\pi}} \int_0^t \frac{z^4(\tau)}{\sqrt{t-\tau}}\, d\tau = T_i + \frac{2T_c(\mathrm{Bi} + \mathrm{Bi_p} T_c^3)}{\sqrt{\pi}} \sqrt{t} + \mathrm{Os}\, q_0 t. \tag{29}$$

The solution of Eq. (29) may be written in the form of an infinite series

$$z(t) = \sum_{n=0}^{\infty} c_n t^{n/2}, \tag{30}$$

where

$$\begin{gathered} c_0 = T_i;\ c_1 = \frac{2\,[T_c(\mathrm{Bi} + \mathrm{Bi_p} T_c^3) - (\mathrm{Bi} T_i + \mathrm{Bi_p} f_0)]}{\sqrt{\pi}}; \\ c_2 = \mathrm{Os}\, q_0 - \frac{\sqrt{\pi}\,(\mathrm{Bi}\, c_1 + \mathrm{Bi_p} \cdot f_1)}{2}; \\ c_{n+1} = -\frac{2^{n-1} n^2 \Gamma^2\left(\frac{n}{2}\right)}{\sqrt{\pi}\,(n+1)!} (\mathrm{Bi} c_n + \mathrm{Bi_p} f_n),\ n = 2, 3, \ldots; \\ f_n = \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{l=0}^{n-m} c_k c_{m-k} c_l c_{n-m-l}. \end{gathered} \tag{31}$$

Substitution of Eq. (31) in Eq. (25) finally gives the temperature distribution in a semiinfinite immobile medium radiating into the surrounding medium by Newton's and the Stefan–Boltzmann laws:

$$\begin{aligned} T(x, t) = T_0(x, t) = T_i &+ \frac{2T_c(\mathrm{Bi} + \mathrm{Bi_p} T_c^3)}{\sqrt{\pi}} \sqrt{t} \exp\left(-\frac{x}{4t}\right) + \mathrm{Os}\, q_0 t - \\ &- \frac{2T_c(\mathrm{Bi} + \mathrm{Bi_p} T_c^3)}{\sqrt{\pi}} x \,\mathrm{Erfc}\left(\frac{x}{2\sqrt{t}}\right) - \frac{\exp\left(-\frac{x}{4t}\right)}{\pi} \sum_{n=0}^{\infty} (\mathrm{Bi} c_n + \mathrm{Bi_p} f_n)\, \Gamma\left(\frac{n+2}{2}\right) t^{\frac{n+1}{2}} G_1\left(\frac{n+2}{2}, \frac{1}{2}, \frac{x^2}{4t}\right), \end{aligned} \tag{32}$$

where Erfc(z) is the complementary minor of the error function, $\mathrm{Erfc}(z) = \int_z^\infty \exp(-s^2)\, ds$; $G_1(\alpha, \gamma, z)$, is a degenerate hypergeometric function of the second sort,

$$G_1(\alpha, \gamma, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp(-zt)\, t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt;$$

and $\Gamma(z)$ is the Euler gamma function.

If the semiinfinite body radiates only according to Newton's law, then the series in Eq. (32) is summed and for T(x, t) we obtain a closed formula.

NOTATION

t, time; M, point in space; $\rho(M, t)$, density of medium; p(M, t), pressure; $\mathbf{V}(M, t)$, velocity of medium; T(M, t), temperature of medium; $a(M, t) = \lambda/C_p\rho(M, t)$, thermal diffusivity of medium; λ, thermal conductivity of medium; C_p, specific heat; $\alpha(M, t)$, coefficient of absorption; σ_0, Stefan–Boltzmann constant; A(M, t), absorptive capability of boundary surface; V, volume occupied by medium; μ, dynamic viscosity coefficient; q, efficiency of internal heat sources (sinks); Diss $f(\mathbf{V})$, dissipative Rayleigh function; $T_i(M)$, initial temperature; $T_W(M, t)$, temperature of boundary surface; $T_c(M, t)$, temperature of surrounding medium; α_{em}, heat emission coefficient; z(t) = T(0, t), temperature at boundary of semiinfinite body; $\Gamma(z)$ Euler gamma function; $G_1(\alpha, \gamma, z)$, degenerate hypergeometric function of the second sort; Erfc (z), complementary minor of error function.

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MATHEMATICAL MODELING OF RADIANT HEAT-EXCHANGE PROCESSES IN METALLURGICAL THERMOTECHNOLOGY

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Various models of the heat-exchange process in metallurgical furnaces are considered: old methods of calculation, the zonal method, and mathematical models of radient and complex heat exchange in plane and cylindrical channels.

A new concept of the "mathematical model" has appeared in contemporary scientific literature. The creation of a mathematical model must be preceded by development of a physical picture of the phenomena (physical model) which determines the geometry of the system and peculiarities of the processes described by the mathematical model, let us say the character of the motion of the medium (stack gases), the values of physical constants, the rate of fuel burnup, etc. These processes are presented in a simplified manner, since an accurate description of the operation of the aggregate is impossible. This is the first stage of the problem. The second is the composition of equations describing the processes and boundary conditions. This is the mathematical modeling. The third stage is the solution of these equations, i.e., obtaining concrete results for model (furnace) operation. The three stages together comprise a method for calculating heat transfer in a furnace. Finally, the last step is adaptation of the model to the actual aggregate, i.e., verification of the calculation with experimental data with subsequent correction, bringing to life, as it were, the mathematical model developed.

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